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SOME APPLICATIONS OF THE GREENS' FUNCTION METHOD IN MECHANICS

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Abstract—Almost each applied mechanical question leads to some boundary value problem of mathematical physics for the body of a complicated shape, whose boundary surface does not coincide with the coordinate surfaces of the selected coordinate system. Mathematical physics, however, has traditionally developed methods (separation of variables, variational, integral transformations, etc.) which are basically suitable for bodies bounded by simple coordinate surfaces.

The finite differences technique applied to problems for bodies of complicated shape requires too large arrays for the storing of interior data so it cannot compete with the widely used method of finite elements.

The basic idea of the Boundary-integral equation (BIE) method is to represent the unknown solution of the given problem in terms of the surface integral, the kernel of which is the fundamental solution of the governing operator. This method recently became rather popular in mechanics because it has some known[13] advantages in comparison with the finite element analysis.

The main distinction between the potential approach which is described in our paper and the BIE method is that here instead of the fundamental solution for the kernel of the potential we are using the Green's matrix of some domain for which the given one is only some portion. Therefore corresponding potentials satisfy the boundary conditions on the part of the bounding surface automatically. Subsequently, we are left only with the necessity of treating the remaining part of the boundary. This approach allows us to solve the problems[2–6, 14–18] for bodies of very complicated geometry.

The first section of the paper contains an explanation of the basic idea of the approach. The numerical examples which are presented in this section applied to the elastic torsion problem.

Two- and three-dimensional steady-state heat conduction problems are presented in the second section. There one can find, for instance, the case of the layered strip with arbitrary holes.

The last section contains a description of the algorithm for constructing of the Green's matrix for the sandwich type of elastic body. Some numerical examples for the homogeneous and layered strip with holes of various shapes are shown.

Other possible applications of the potential method described are briefly reviewed in the conclusion of the paper.

1. THE BASIC IDEA OF THE APPROACH

Consider a case of the elastic torsion of the prizmatic bar. This simple mechanical problem is chosen in order to introduce the main ideas of the general approach to be used. As it is well known, the governing equations for the torsion of a prizmatic bar of multiconnected cross-section $L = \bigcup_{i=0}^{n} L_i$ (L_0–the outer contour) can be written as

$$\nabla^2 U(x_1, x_2) = -2 \tag{1.1}$$

$$U/L_i = C_i \qquad (i = 0, 1, 2, \dots, n) \tag{1.2}$$

where $U = U(x_1, x_2)$ presents a stress function, and one of the constants C_i may be fixed arbitrarily, the remaining constants then being determined by enforcing the condition of single-valuedness on the function $U(x_1, x_2)$.

For the simple connected domain, Ω problem (1.1) and (1.2) can be formulated as an internal Dirichlet problem for the stress function which vanishes on the boundary L_0, i.e.

$$U/L_0 = 0. \tag{1.3}$$

A simple way to represent the solution of the problem (1.1) and (1.3) by means of the classical (see, e.g. [1]) harmonic single-layer potential, is to express the stress function as:

$$U(x_1, x_2) = v(x_1, x_2) + w(x_1, x_2) \tag{1.4}$$

where

$$v(x_1, x_2) = 2\int\int_{\Omega} ln\, r \, d\Omega(\xi_1, \xi_2) \quad (1.5)$$

$$w(x_1, x_2) = \int_{L_0} ln\, r\mu(\xi_1, \xi_2)\, dL_0(\xi_1, \xi_2) \quad (1.6)$$

and

$$r = \sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}.$$

Here, $\mu = \mu(\xi_1, \xi_2)$ is the unknown continuous function (density of the potential (1.6)) which may be determined by satisfying condition (1.3), point (x_1, x_2) belongs to the domain Ω and point (ξ_1, ξ_2) belongs to the boundary L_0.

Insofar as lnr is the principal solution of the Laplace equation, eqns (1.1) and (1.3) will be satisfied by the solution (1.4) provided the integral equation

$$-v_{L_0}(x_1, x_2) = \int_{L_0} ln\, r\mu(\xi_1, \xi_2)\, dL_0(\xi_1, \xi_2) \quad (1.7)$$

is solved. Here, $v_{L_0}(x_1, x_2)$—values of the integral (1.5) on the line L_0—can be accurately obtained by any one of several rules of approximate integration. The integral eqn (1.7) may then be numerically integrated since the kernel lnr has an integrable singularity.

The above approach to solving such mechanical problems is well known, e.g. Ref. [13]. But sometimes this approach can be improved so as to lead to a more compact numerical procedure.

Thus, we begin by assuming the contour L_0 to be divided in two parts, Λ_0 and Γ, in such a way that Green's function $g(x_1, x_2; \xi_1, \xi_2)$† is known for the domain Ω' (see Fig. 1), for which Ω is some portion.

The solution of problem (1.1), (1.3) as given by (1.4) may then be re-written as

$$v(x_1, x_2) = 2\int\int_{\Omega'} g(x_1, x_2; \xi_1, \xi_2)\, d\Omega'(\xi_1, \xi_2) \quad (1.8)$$

and

$$w(x_1, x_2) = \int_{\Lambda} g(x_1, x_2; \xi_1, \xi_2)\mu(\xi_1, \xi_2)\, d\Lambda(\xi_1, \xi_2) \quad (1.9)$$

where $\mu(\xi_1, \xi_2)$ once again is the unknown continuous function to be determined by satisfaction of the boundary condition (1.3).

It is to be noted that since eqns (1.8) and (1.9) contain Green's function as their kernels, the solution (1.4) vanishes on the line Γ. Upon invoking the Boundary condition (1.3), we are then led to the following integral equation at points (x_1, x_2) that lie on the Λ:

$$-v_{\Lambda}(x_1, x_2) = \int_{\Lambda} g(x_1, x_2; \xi_1, \xi_2)\mu(\xi_1, \xi_2)\, d\Lambda(\xi_1, \xi_2). \quad (1.10)$$

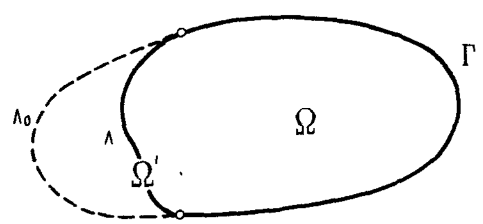


Fig. 1.

†We will consider here Green's function of the Dirichlet problem.

Here, $v_A(x_1, x_2)$—values of the integral (1.8) on the line Λ—can be computed by any rule of approximate integration.

Upon solving this equation for the potential density function $\mu(\xi_1, \xi_2)$, the solution of the original problem obtains.

Figure 2 shows some results calculated by the algorithm described above. The given domain here is a square whose sides are equal to $2a$ and with quarter-circle, corner cut-outs of radius R. The Green's function for the square was employed as the kernel of integrals (1.8) and (1.9).

The maximum shear stresses τ are shown in Fig. 2(b) for $R/A = 0.5$. Appearing on the left of this figure are the maximum shear stresses which develop on the contour of the given cross-section; on the right appear those stresses acting at internal points of the domain. Variation of the maximum shear stresses τ_A and τ_B with the ratio R/a at the two contour points A and B of greatest interest are shown in Fig. 2(c). It should be noted that $\tau_A = \tau_B$ when the $R/a = 0.31$; thus, there are eight points on the contour of this cross-section which have equal maximum shear stresses. Moreover, the shear stress τ_A has its maximum value when $R/a = 0.47$ slightly larger than the maximum value of τ_B which occurs when $R/a = 0$ (square without cut-outs).

The approach described may also be used for multiple connected cross-sections, it is convenient even for compound cross-sections. Figure 3, for example, shows the distribution of the maximum shear stresses appearing on the contour of the compound rectangular cross-section with elliptical cut-outs whose boundaries cut the interface between two materials. The upper material here is half as stiff as the lower one.

2. POTENTIAL METHOD APPLIED TO STEADY-STATE HEAT CONDUCTION PROBLEMS

A variant of the Green's function method is used here to solve steady-state heat conduction problems in multiple-connected bodies involving complicated geometries. An algorithm for constructing the Green's matrix of a layered strip is described and numerical results for two- and three-dimensional particular cases are presented. This approach is general for systems of elliptical partial differential equations in mechanics and it has been employed in the past to obtain numerical solutions of various mechanical problems for bodies with intricate shapes.

The particular problems considered here will be used merely as examples to demonstrate the application of the method for steady-state heat conduction boundary value problems.

Statement of the problem

Consider a two-dimensional, n-layered strip with an arbitrary hole L, and assume that the

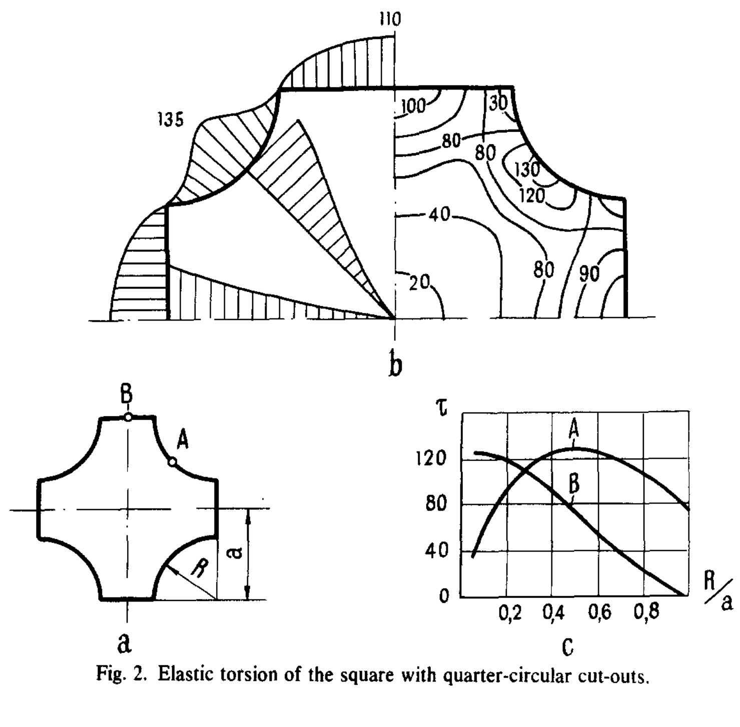


Fig. 2. Elastic torsion of the square with quarter-circular cut-outs.

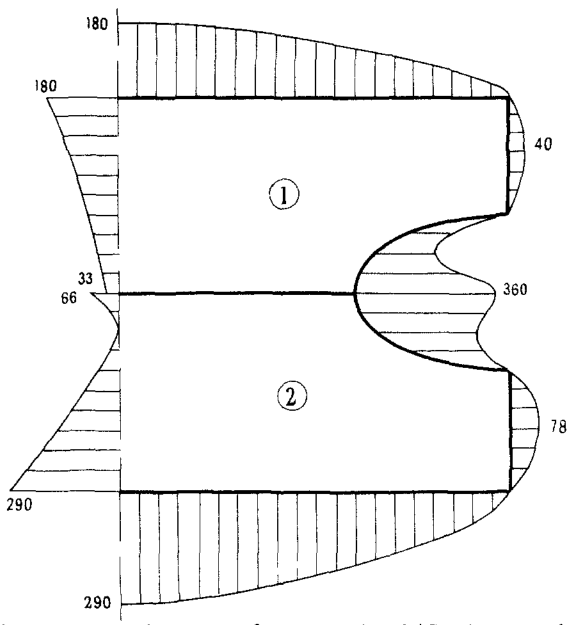


Fig. 3. Maximum shear stresses on the contour of the composite ($G_2/G_1 = 2$) rectangular cross-section with two elliptical cut-outs.

thermal conductivities λ_k ($k = 1, 2, \ldots, n$) in each of the n layers are known continuous functions of the coordinate x (see Fig. 4).

The temperature u in the strip will satisfy the following relations:

$$\frac{\partial}{\partial x}\left(\lambda_k(x)\frac{\partial u_k}{\partial x}\right) + \lambda_k(x)\frac{\partial^2 u_k}{\partial y^2} = f_k(x, y) \qquad (k = 1, 2, \ldots, n) \tag{2.1}$$

$$A_1 u_1|_{x=a_0} = 0; \qquad A_2 u_n|_{x=a_n} = 0; \tag{2.2}$$

$$u_k|_{x=a_k} = u_{k+1}|_{x=a_k}, \quad \lambda_k \frac{\partial u_k}{\partial x}\bigg|_{x=a_k} = \lambda_{k+1}\frac{\partial u_{k+1}}{\partial x}\bigg|_{x=a_k} \tag{2.3}$$

$$\frac{\partial u_k}{\partial y}\bigg|_{y=\pm\infty} = 0, \tag{2.4}$$

$$B u_k|_L = Q(x, y) \tag{2.5}$$

where $u_k = u_k(x, y)$ is the unknown temperature of the k-th layer, A_1, A_2 and B are linear differential operators that depend on the boundary conditions, and $Q(x, y)$ and $f_k(x, y)$ are given continuous functions. In the next section the Green's matrix for the relations (2.1)–(2.4) will be constructed in order to obtain the solution of the problem (2.1)–(2.5) by means of an integral representation in which the Green's matrix is the kernel.

Construction of the Green's matrix

It will be assumed that problem (2.1)–(2.5) has symmetry about the x-axis and that the functions $u_k(x, y)$ and $f_k(x, y)$ can be represented in terms of Fourier integrals

$$u_k(x, y) = \frac{1}{\pi}\int_0^\infty U_k(x, \omega)\cos \omega y \, d\omega; \quad f_k(x, y) = \frac{1}{\pi}\int_0^\infty F_k(x, \omega)\cos \omega y \, d\omega. \tag{2.6}$$

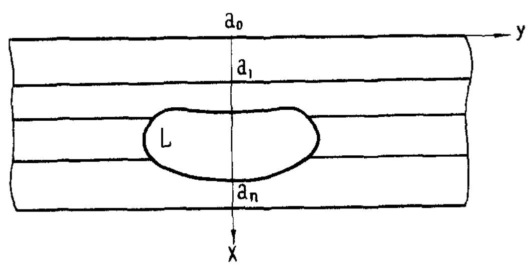


Fig. 4.

Substitution of (2.6) into (2.1)–(2.4) leads to the following system of ordinary differential equations.

$$\lambda_k \frac{d^2 U_k(x, \omega)}{dx^2} + \frac{d\lambda_k}{dx}\frac{dU_k(x, \omega)}{dx} - \omega^2 \lambda_k U_k(x, \omega) = F_k(x, \omega) \tag{2.7}$$

with the boundary conditions

$$A_{1\omega} U_1|_{x=a_0} = 0, \qquad A_{2\omega} U_n|_{x=a_n} = 0 \tag{2.8}$$

$$U_k|_{x=a_k} = U_{k+1}|_{x=a_k}; \quad \lambda_k \frac{dU_k}{dx}\bigg|_{x=a_k} = \lambda_{k+1} \frac{dU_{k+1}}{dx}\bigg|_{x=a_k}. \tag{2.9}$$

A set of linearly independent particular solutions of the homogeneous system corresponding to eqns (2.7) can be obtained numerically by any suitable method (as, for instance, Euler or Runge–Kutta). The Green's matrix for the relations (2.7)–(2.9)

$$g(x, \xi; \omega) = (g_{ij}(x, \xi; \omega))_{i,j=1}^{n} \tag{2.10}$$

may be obtained from this set of linearly independent particular solutions by employing a known procedure as, for example, that described and used in references [2, 5, 16]. For n layers in the strip the order of the Green's matrix is also n. On this basis the solution of (2.7)–(2.9) may be expressed as

$$U(x, \omega) = \int_{a_0}^{a_n} g(x, \xi; \omega) F(\xi, \omega)\, d\xi$$

where $U(x, \omega)$ and $F(\xi, \omega)$ are vectors with components $U_k(x, \omega)$ and $F_k(\xi, \omega)$, respectively.

Using the inverse Fourier transformation for the last equation

$$F(\xi, \omega) = 2 \int_0^{\infty} f(\xi, \eta) \cos \omega\eta \, d\eta$$

and substituting the resulting expression into the first of (2.6) and then reversing the order of integrations yields

$$u(x, y) = \int_0^{\infty} \int_{a_0}^{a_n} \left[\frac{2}{\pi} \int_0^{\infty} g(x, \xi; \omega) \cos \omega y \cos \omega\eta \, d\omega\right] f(\xi, \eta)\, d\xi\, d\eta \tag{2.11}$$

where $u(x, y)$ and $f(\xi, \eta)$ are vectors with components $u_k(x, y)$ and $f_k(\xi, \eta)$, respectively.

Assuming that the solutions of (2.1)–(2.4) are unique, the kernel of the double integral in (2.11) is the Green's matrix $G(x, y; \xi\eta)$ of the given problem. Thus

$$G(x, y; \xi, \eta) = \frac{2}{\pi} \int_0^{\infty} g(x, \xi; \omega) \cos \omega y \cos \omega\eta \, d\omega.$$

Algorithm of computation and numerical examples

Consider for example, the case in which the operator B is unity, meaning that the temperature is specified on the contour L. Represent the solution of (2.1)–(2.5) by the following sum [16]

$$u(x, y) = v(x, y) + w(x, y) \tag{2.13}$$

here

$$v(x, y) = \int_0^{\infty} \int_{a_0}^{a_n} G(x, y; \xi, \eta) f(\xi, \eta)\, d\xi\, d\eta \tag{2.14}$$

and

$$w(x, y) = \int_L G(x, y; \xi, \eta) h(\xi, \eta)\, dL(\xi, \eta). \tag{2.15}$$

The kernel $G(x, y;\ \xi, \eta)$ of the representations above is the Green's matrix described in the previous section. Function $h(\xi, \eta)$ is the unknown density of the potential (2.15), which can be determined from the requirement that (2.13) satisfies the boundary conditions (2.5). On this basis we obtain

$$Q(x, y) - \tilde{v}(x, y) = \int_L G(x, y; \xi, \eta) h(\xi, \eta)\, dL(\xi, \eta) \tag{2.16}$$

which is a Fredholm integral equation for $h(\xi, \eta)$. The term $\tilde{v}(x, y)$ is equal to the value of the integral (2.14) on the line L. After the integral equation is solved for the potential density $h(\xi, \eta)$, the solution of (2.1)–(2.5) is immediately obtained from (2.13). The described algorithm was applied to a two-layered strip with a tunnel-shaped hole for the following data: $f(x, y) \equiv 0$, $Q(x, y) \equiv 1$, $B = A_1 \equiv 1$ and $A_2 \equiv \partial/\partial_x + 1$. Figure 5(a) gives the variation of the thermal conductivities of the given materials transversely to the layers.

The previously mentioned set of the linearly independent particular solutions of the homogeneous system (2.7) can in this case be obtained in closed form because of the exponential representations of the λ_k coefficients. The integral eqn (2.16) was solved by expressing the integrals as finite sums by means of the trapezoidal rule. The locations and spacing of the grid points must be generally determined by taking into account the size and shape of the hole L. In this example we used a set of 24 uniformly spaced grid points. The improper integral in (2.12) converges rather fast, and it could be calculated with sufficient accuracy without any difficulties. The resulting temperature field for the case considered is shown in Fig. 5(b).

The three-dimensional case of a homogeneous layer with the tunnel S of an elliptical cross-section is shown in Fig. 6. The boundary conditions here are specified as

$$u|_{x=0} = 0, \quad \left(\frac{\partial u}{\partial x} + u\right)_{x=1} = 0, \quad u|_s = \cos^{32} z$$

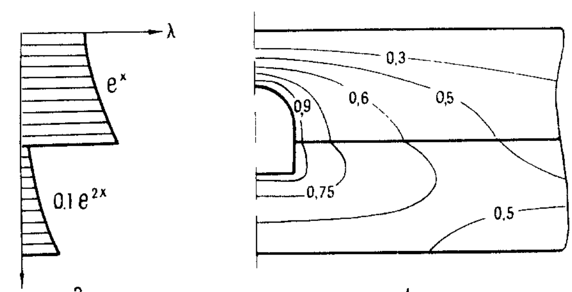


Fig. 5. Temperature field of the two-layered strip with a tunnel-shaped hole.

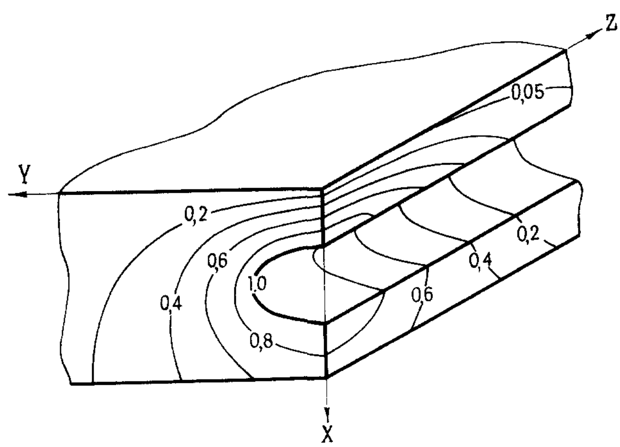


Fig. 6. Three-dimensional problem of the steady-state heat conduction for the layer with an elliptical tunnel.

Instead of the artificial boundary conditions on the surface S one could specify any condition that is more realistic from a physical point of view. The artificial condition was selected here only for the purpose of making the example simpler. The three-dimensional problem considered was reduced to a two-dimensional problem by means of Fourier series in the z direction, and subsequently the algorithm described above was employed. From the standpoint of calculations such an approach to solve three-dimensional problems does not require much additional work except for computer programming.

3. APPLICATION TO THE PROBLEMS OF PLANE THEORY OF ELASTICITY

The Green's matrix approach described in the previous sections is used here to obtain the solutions of two-dimensional elasticity problems of an infinite strip with periodically spaced holes. Numerical results are presented and generalizations to the case of a layered strip obtained.

Attempts to obtain elasticity solutions of boundary value problems for bodies of complicated shapes usually result in great numerical difficulties, which can be successfully overcome only for particular situations.

Let us consider a domain in which complexity arises because part of the bounding surfaces do not coincide with the coordinate surfaces of the selected coordinate system. Such a situation occurs, for example, in the plane elasticity problem of a strip with holes, the case in which the holes are circular having been solved in Ref. [7] by using the technique of conformal mapping. Numerical procedures, like the widely used finite element method, can also be employed to solve such problems.

In this section of our paper, use is made of the Green's matrix method, which reduces the problem to the solution of Fredholm's integral equations. The procedure is similar to that employed in the previous sections and successfully competes with finite element analysis in terms of computer time and accuracy of the final results.

Statement of the problem

Consider an elastic strip, with a row of periodically spaced arbitrary holes Γ (see Fig. 7), resting on a frictionless rigid foundation. Assume that traction boundary conditions are prescribed on the surface $x = 0$ and on the contours Γ. It is quite possible to consider any other set of boundary conditions, but this detail will be described later.

The displacement formulation of the boundary-value problem in the region Ω (see Fig. 7) follows:

$$L\left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \lambda, \mu\right)U(x, y) = \bar{F}(x, y) \tag{3.1}$$

$$B_1\left(\frac{\partial}{\partial x}\right)U(a, y) = \bar{\Phi}_1(y)$$

$$B_2\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \lambda, \mu\right)U(0, y) = \bar{\Phi}_2(y) \tag{3.2}$$

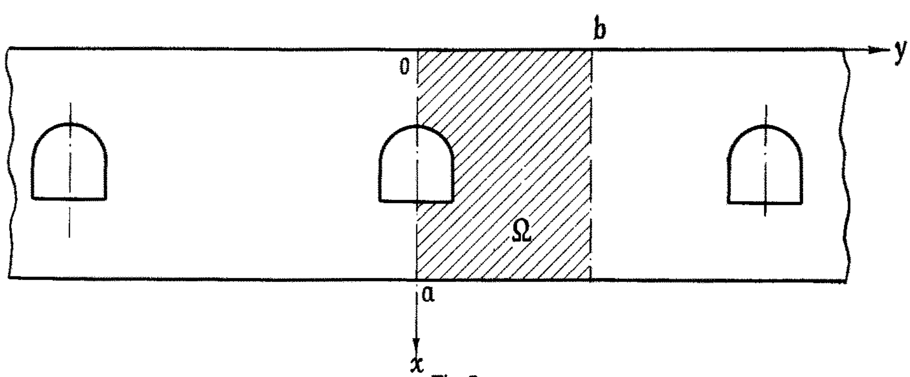


Fig. 7.

$$B_3\left(\frac{\partial}{\partial y}\right)U(x, \pm b) = 0 \tag{3.3}$$

$$B\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \lambda, \mu\right)U|_\Gamma = \tilde{\Phi}(x, y) \tag{3.4}$$

where $U(x, y)$ and $\tilde{F}(x, y)$ are the displacement and body force vectors, respectively; the matrix–operator L is

$$L \equiv \begin{pmatrix} (\lambda+\mu)\dfrac{\partial^2}{\partial x^2} + \nabla^2 & (\lambda+\mu)\dfrac{\partial^2}{\partial x \partial y} \\ (\lambda+\mu)\dfrac{\partial^2}{\partial x \partial y} & (\lambda+\mu)\dfrac{\partial^2}{\partial y^2} + \nabla^2 \end{pmatrix}, \tag{3.5}$$

λ, μ are the Lamé constants; ∇^2 is Laplace's operator. According to our assumptions the column $\tilde{\Phi}_1(y)$ must be equal to zero, and the operators B_1, B_2 and B_3 can be written as follows

$$B_1 \equiv \begin{pmatrix} 1 & 0 \\ \mu\dfrac{\partial}{\partial y} & \mu\dfrac{\partial}{\partial x} \end{pmatrix}; \quad B_2 \equiv \begin{pmatrix} (\lambda+2\mu)\dfrac{\partial}{\partial x} & \lambda\dfrac{\partial}{\partial y} \\ \mu\dfrac{\partial}{\partial y} & \mu\dfrac{\partial}{\partial x} \end{pmatrix}$$

$$B_3 \equiv \begin{pmatrix} \dfrac{\partial}{\partial y} & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.6}$$

Dividing equations (3.1) and boundary conditions (3.2) and (3.4) by μ and letting $\sigma = 1 + \lambda/\mu$, we obtain the following

$$L\left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{dy^2}, \sigma\right) = F(x, y) \tag{3.7}$$

$$B_1\left(\frac{\partial}{\partial x}\right)U(a, y) = 0 \tag{3.8}$$

$$B_2\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \sigma\right)U(0, y) = \Phi_2(y) \tag{3.9}$$

$$B_3\left(\frac{\partial}{\partial y}\right)U(x, \pm b) = 0 \tag{3.10}$$

$$B\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \sigma\right)U|_\Gamma = \Phi(x, y). \tag{3.11}$$

It is necessary to note that the left-hand sides of the above contain only one parameter σ instead of two parameters λ, μ in the preceding formulation. This proves to be very convenient for a parametric study.

Constructing the Green's matrix

Consider now eqns (3.7), boundary conditions (3.8), (3.10) and the homogeneous condition corresponding to (3.9)

$$B_2\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \sigma\right)U(0, y) = 0. \tag{3.12}$$

We now proceed to construct Green's matrix of (3.7), (3.8), (3.10) and (3.12) for the rectangle region Ω. For this purpose, represent vectors $U(x, y)$ and $F(x, y)$ by the series

$$U(x, y) = \sum_{n=0}^{\infty} Q_n(y)U_n(x); \quad F(x, y) = \sum_{n=0}^{\infty} Q_n(y)F_n(x) \tag{3.13}$$

where

$$Q_n(y) = \begin{pmatrix} \cos \nu y & 0 \\ 0 & \sin \nu y \end{pmatrix} \quad \text{and} \quad \nu = \frac{n\pi}{b} \tag{3.14}$$

It is quite clear that such a representation of the vector $U(x, y)$ satisfies the boundary conditions of eqn (3.10). Substituting (3.13) into (3.7), (3.8), (3.12), we obtain the system of ordinary differential equations

$$L_n\left(\frac{d^2}{dx^2}, \sigma\right)U_n(x) = F_n(x); \quad (n = 0, 1, 2 \ldots) \tag{3.15}$$

with boundary conditions

$$B_{1n}\left(\frac{d}{dx}\right)U_n(a) = 0 \tag{3.16}$$

$$B_{2n}\left(\frac{d}{dx}, \sigma\right)U(0) = 0 \tag{3.17}$$

Here

$$L_n\left(\frac{d^2}{dx^2}, \sigma\right) \equiv \begin{pmatrix} (\sigma+1)\frac{d^2}{dx^2} - \nu^2 & \sigma\nu\frac{d}{dx} \\ -\sigma\nu\frac{d}{dx} & \frac{d^2}{dx^2} - (\sigma+1)\nu^2 \end{pmatrix};$$

$$B_{1n}\left(\frac{d}{dx}\right) \equiv \begin{pmatrix} 1 & 0 \\ -\nu & \frac{d}{dx} \end{pmatrix}; \quad B_{2n}\left(\frac{d}{dx}, \sigma\right) \equiv \begin{pmatrix} (\sigma+1)\frac{d}{dx} & (\sigma-1)\nu \\ -\nu & \frac{d}{dx} \end{pmatrix} \tag{3.18}$$

It is not difficult to show that the vectors

$$U_{n1}(x) = \begin{pmatrix} \cosh \nu x \\ -\sinh \nu x \end{pmatrix}; \quad U_{n3}(x) = \begin{pmatrix} -\sigma\nu x \sinh \nu x \\ \sigma\nu \cosh \nu x + (\sigma+2)\sinh \nu x \end{pmatrix}$$
$$U_{n2}(x) = \begin{pmatrix} \sinh \nu x \\ -\cosh \nu x \end{pmatrix}; \quad U_{n4}(x) = \begin{pmatrix} -\sigma\nu x \cosh \nu x \\ \sigma\nu x \sinh \nu x + (\sigma+2)\cosh \nu x \end{pmatrix} \tag{3.19}$$

are linearly independent solutions of the homogeneous system of eqns (3.15). Consequently, the general solution of eqns (3.15) can be written as

$$U_n(x) = P_n(x) \cdot C_n(x) \tag{3.20}$$

where $P_n = (U_{nj}(x))$ is a 2×4 matrix whose columns are the vectors of eqn (3.19) and $C_n(x)$ is a column of unknown functions.

In accordance with Lagrange's method of variation of arbitrary constants, we obtain the system of linear equations,

$$P_n^*(x) \cdot C_n'(x) = F_n^*(x) \tag{3.21}$$

where

$$P_n^*(x) = \begin{pmatrix} P_n(x) \\ P_n'(x) \end{pmatrix} \quad \text{and} \quad F_n^*(x) = \begin{pmatrix} 0 \\ F_n(x) \end{pmatrix}. \tag{3.22}$$

Since $P_n^*(x)$ is nonsingular,

$$C_n'(x) = (P_n^*(x))^{-1} \cdot F_n^*(x)$$

or

$$C_n'(x) = R_n(x) \cdot F_n(x) \tag{3.23}$$

where $R_n(x)$ is a 4×2 matrix whose columns are the third and fourth columns of the matrix $(P_n^*(x))^{-1}$.

Integrating (3.23) gives us

$$C_n(x) = \int_0^x R_n(\xi) F_n(\xi)\, \mathrm{d}\xi + D_n \tag{3.24}$$

Upon substituting this expression into (3.20), we obtain

$$U_n(x) = \int_0^x S_n(x, \xi) F_n(\xi)\, \mathrm{d}\xi + P_n(x) D_n \tag{3.25}$$

where

$$S_n(x, \xi) = P_n(x) \cdot R_n(\xi). \tag{3.26}$$

Hence, when satisfying the boundary conditions (3.16), (3.17) we obtain the following system of linear algebraic equations for the vector D_n

$$T_n \cdot D_n = \psi_n \tag{3.27}$$

The first and the second rows of the matrix T_n are given by the expression

$$B_{2n}\left(\frac{\mathrm{d}}{\mathrm{d}x}, \sigma\right) P_n(0),$$

the third and the fourth rows—by the expression

$$B_{1n}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) P_n(a)$$

The first and the second elements of the column ψ_n are zero, and the third and the fourth are given by

$$\int_0^a Z_n(a, \xi) F_n(\xi)\, \mathrm{d}\xi$$

where

$$Z_n(x, \xi) = -B_{1n}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) S_n(x, \xi).$$

It can thus be seen that one can readily consider boundary conditions other than those given by (3.8) and (3.12). Only the expressions for the elements of the matrices T_n and ψ_n would be affected. The algorithm described here remains unchanged, and is independent of the boundary conditions on the edges $x = 0$, a.

Let us now write down the solution of the system (3.27) as follows

$$D_n = \int_0^a W_n(\xi)F_n(\xi)\,d\xi \tag{3.28}$$

where $W_n(\xi) = T_n^{-1}Z_n(a, \xi)$ is a 4×2 matrix. Substituting this expression for the vector D_n into (3.25) we obtain

$$U_n(x) = \int_0^x S_n(x, \xi)F_n(\xi)\,d\xi + \int_0^a P_n(x)W_n(\xi)F(\xi)\,d\xi \tag{3.29}$$

or

$$U_n(x) = \int_0^a g_n(x, \xi)F_n(\xi)\,d\xi \tag{3.30}$$

where

$$g_n(x, \xi) = \begin{cases} S_n(x, \xi) + P_n(x)W_n(\xi) & \text{for} \quad x \geq \xi \\ P_n(x)W_n(\xi) & \text{for} \quad x \leq \xi \end{cases}.$$

If we now express $F_n(\xi)$ by the Fourier's–Euler's Expansions in accordance with (3.13) and substitute (3.30) into the first of (3.13) it is possible to express the solution of the problem defined by eqns (3.7), (3.8), (3.10) and (3.12) as follows

$$U = \int_0^a \int_0^b \left[\frac{\epsilon_n}{b}\sum_{n=0}^{\infty} Q_n(y)g_n(x, \xi)Q_n(\eta)\right]F(\xi, \eta)\,d\xi\,d\eta$$

or

$$U(x, y) = \int\int_\Omega G(x, y; \xi, \eta)F(\xi, \eta)\,d\Omega(\xi, \eta) \tag{3.31}$$

where the kernal $G(x, y, \xi, \eta)$ of this representation is Green's matrix for the operator L in the region Ω, and the Neumann factor $\epsilon_n = \begin{Bmatrix} 1, n = 0 \\ 2, n > 0 \end{Bmatrix}$.

Description of the algorithm and generalizations

Let us now consider problem (3.7)–(3.11), and as an example assume that the operator B is unity, that is, the displacements are specified on the contour Γ.

It is quite clear that we can consider the boundary condition (3.12) instead of (3.9) since it is always possible to reduce such a problem to one described by eqns (3.7), (3.8), (3.10), (3.11) and (3.12) (which we will refer to as Problem (A)) by subtraction of some particular solution. We seek (see, for instance, [16]) the solution to Problem (A) in the following form

$$U(x, y) = U_1(x, y) + U_2(x, y). \tag{3.32}$$

$U_1(x, y)$ is defined in (3.31) and $U_2(x, y)$ can be written as

$$U_2(x, y) = \int_\Gamma G(x, y, \xi, \eta)\mu(\xi, \eta)\,d\Gamma(\xi, \eta) \tag{3.33}$$

where $\mu(\xi, \eta)$ is the potential density function.

The functions of eqn (3.32) satisfies (3.7), (3.8), (3.10) and (3.12), since the kernels of (3.31) and (3.33) are Green's matrix for the operator L in the region Ω. Finally, upon invoking boundary conditions (3.11), we are led to the following system of integral equations at points (x, y) that lie on Γ,

$$\int_\Gamma G(x, y, \xi, \eta)\mu(\xi, \eta)\,d\Gamma(\xi, \eta) = \Phi(x, y) - U_1(x, y). \tag{3.34}$$

Upon solving these equations for the potential density function $\mu(\xi, \eta)$, we immediately obtain the solution of Problem (A) from (3.32).

Equation (3.34) is solved by expressing the integrals as finite sums by means of the trapezoidal rule. The locations and spacing of the grid points must be determined by considering the size and shape of the hole Γ. In our first example we used a set of 24 points. For Green's matrix we employed ten terms of the corresponding Fourier series.

It has been determined from detailed calculations, that further increasing the amount of grid points and the number of terms of the series does not result in any significant change in the final results. Thus, we conclude that the convergence of the series in (31)–(34) is sufficiently fast.

Such an algorithm can be generalized for the nonhomogeneous strip with holes, if the elastic material properties are functions of the x variable. Nothing in the algorithm has to be changed except for the vectors (3.19). In the general case these vectors can be obtained approximately by any (for instance, Euler or Runge–Kutta) method for Couchy's problems for systems of ordinary differential equations. Such an approach had been used in our third example, where we had considered a two-layered sandwich strip.

Numerical examples

As a first example consider the homogeneous isotropic strip with a periodic distribution of two closely-spaced elliptical clamped holes under uniform pressure on the upper edge. In Fig. 8, we show the state of deformation of such a strip.

A more difficult case, is the second example where we considered trapezoidal holes with smooth corners. Such hole shapes can be found, for example, in mining and subway construction, etc. Here contour Γ is free and the upper edge is loaded by a local normal load which is represented by the expression

$$X(y) = X_0 \cos^{2n} \frac{\pi y}{2b}$$

with $n = 32$. We see that such a form approximates a concentrated load. Figure 9(a) shows the resulting deformation state, while the maximum shear stress τ and the principal normal stresses σ_1, σ_2, are shown in Figs. 9(b–d). It is interesting to note the stress concentration in the neighborhood of the smooth corners of the contour Γ and at the point which is situated on the contour Γ exactly under the load point. As expected, we can also see an essentially unstressed zone below the hole.

As a final example, we show the results for the two-layered sandwich strip having the same hole shape. The upper layer is half as stiff as the lower one, the hole is clamped and the upper edge is loaded by a normal uniformly distributed pressure. Some results of this case are given in Fig. 10, where we have plotted the distributions of contour normal and tangent reactions.

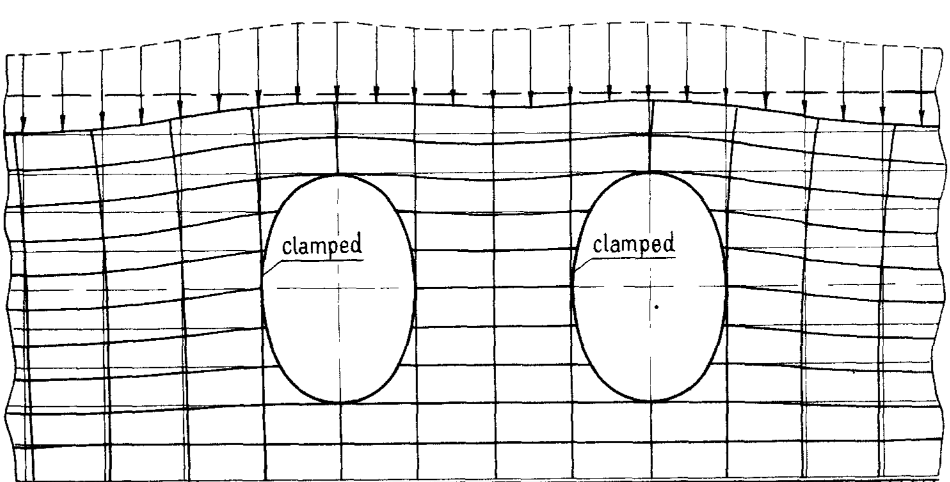
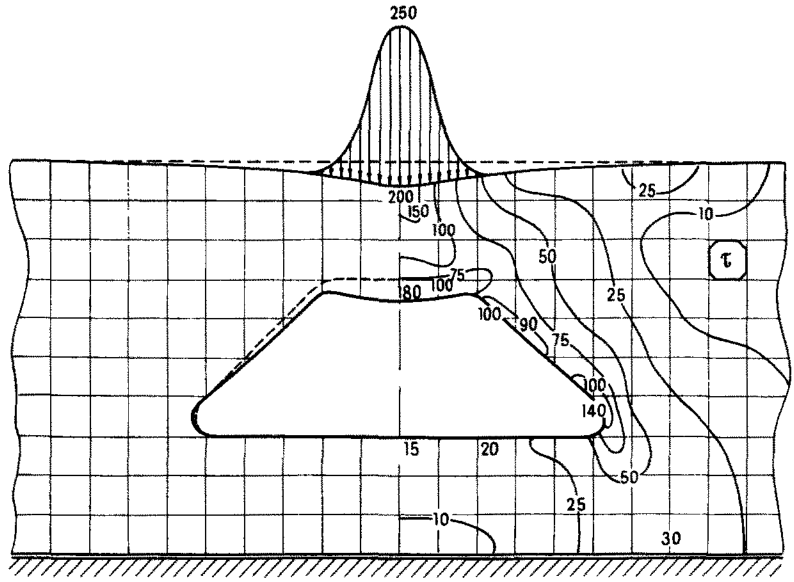
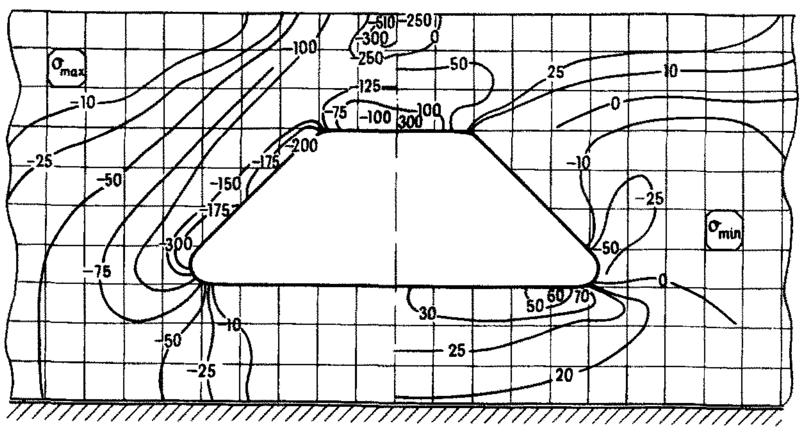


Fig. 8. Deformation field for two closely-spaced elliptical clamped holes.



(a) (b)



(c) (d)

Fig. 9. Principal normal and shearing stresses for trapezoidal hole.

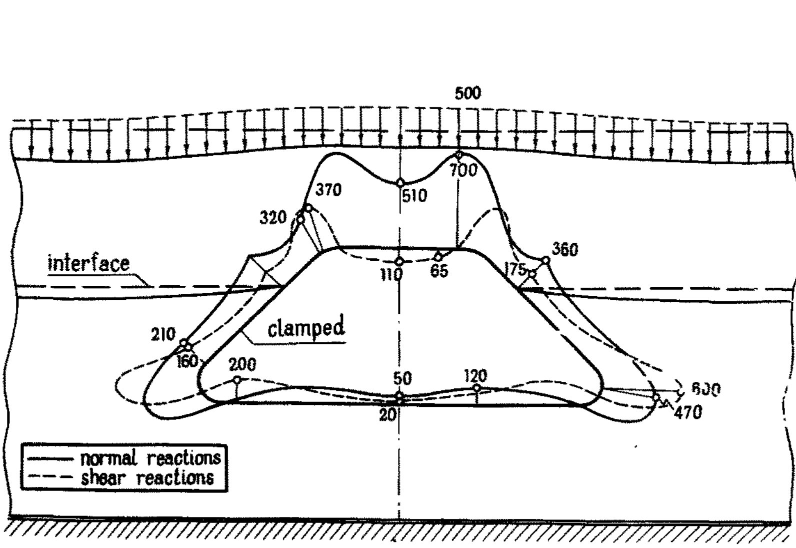


Fig. 10. Layered strip with clamped trapezoidal hole.

Thus, in this section we have demonstrated that our variant of the surface integral method can be successfully applied to solve rather complicated elasticity problems.

Conclusion

In the previous sections of the paper the basic idea of the potential approach has been discussed, the algorithm for construction of the Green's functions (matrices) presented, and the method of solution of some problems of applied mechanics were developed by means of the version of the surface integral method indicated.

It should be noted that the main practical advantage of the technique discussed here is the saving of the computer time. Comparison with the finite element program NASTRAN, which was also used to solve the problem mentioned in the third section showed that the program developed on the basis of the author's approach was ten times faster for the same level of accuracy of the final results.

The variety of possible applications of this method includes many other problems in applied mechanics. Some of them have already been considered by various authors (see, for instance, [2–6, 8, 9, 11, 12]). Possible application of the methods described here include, firstly, the problems of transient heat conduction which could be solved on the basis of potential representation of the unknown functions. Such problems would also include bodies with moving boundaries. A second very interesting group of applications which could be attacked by the approach indicated are vibration problems for compound beams, plates and shells. Additionally, it appears that buckling problems including thermo-buckling phenomena also fall within the range of the techniques presented.

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